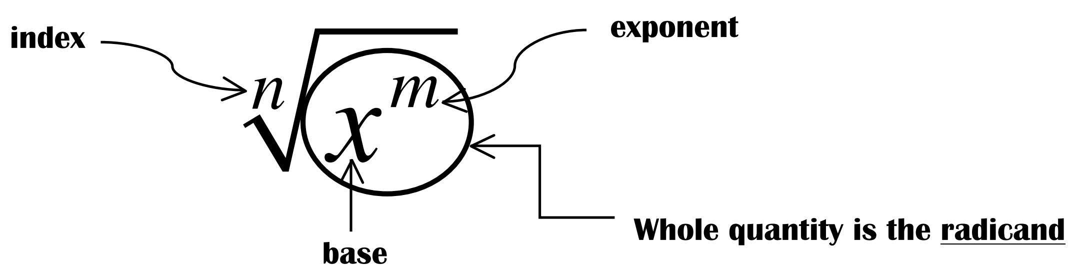
Fractional Exponents

Roots and being comfortable and flexible with them are important to upper classes of mathematics. The earlier we get students comfortable with root extraction the better. In this lesson we will be using three methods of root extraction and simplifying both monomials and polynomial involving roots. We are going to start this lesson with a definition and expand to the mathematic symbols.

Definition: The square root is one of its two equal factors.

This works for any root extraction.

Definition: The cube root is one of its three equal factors
Definition: The fourth root is one of its four equal factors
Definition: The "*x*" root is one of its "*x*" equal factors



Based on this definition the following is true:

$\sqrt[2]{x^2}$	$\sqrt[3]{x^3}$	$\sqrt[4]{x^4}$
$= \sqrt[2]{x \cdot x}$	$= \sqrt[3]{x \cdot x \cdot x}$	$= \sqrt[4]{x \cdot x \cdot x \cdot x}$
$= x$	$= x$	$= x$

Or to simplify:

$$\sqrt[2]{x^2} = x$$

$$\sqrt[3]{x^3} = x$$

$$\sqrt[4]{x^4} = x$$

We see this as inverse operations when the exponent and the index are the same.

Let's recall some exponent properties:

$$x^n \cdot x^m = x^{n+m}$$

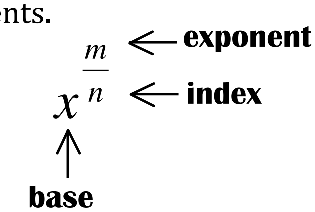
$$\left(x^n\right)^m = x^{n \cdot m}$$

What number could you substitute in the place of *m* to make the equation true?

$$\left(x^2\right)^m = x^1$$

In the above equation, $m = \frac{1}{2}$.

This leads to the concept of rational exponents.



The numerator of a rational exponent denotes the ***exponent*** to which the base is raised, and the denominator denotes the ***index*** or the ***root*** to be taken.

Example 1 Change the expression from radical form to rational exponent form

a. $\sqrt[2]{x^3}$

$= x^{\frac{3}{2}}$

b. $\sqrt[4]{y}$

$= y^{\frac{1}{4}}$

c. You Try

$\sqrt[5]{x^7}$

$= x^{\frac{7}{5}}$

Using the definition, inverse operations and fractional exponents we will now simplify the following expressions.

Example 2

$$\sqrt[3]{24a^4}$$

Definition	Inverse Operation	Fractional Exponent
$\sqrt[3]{24a^4}$ $= \sqrt[3]{2\cdot 2\cdot 2\cdot 3\cdot a\cdot a\cdot a\cdot a}$ $= \sqrt[3]{\underline{2\cdot 2\cdot 2}\cdot 3\cdot \underline{a\cdot a\cdot a}\cdot a}$ $= 2a\sqrt[3]{3a}$	$\sqrt[3]{24a^4}$ $= \sqrt[3]{2^3\cdot 3\cdot a^3\cdot a}$ $= \sqrt[3]{2^3\cdot a^3\cdot 3\cdot a}$ $= \sqrt[3]{2^3\cdot a^3}\cdot\sqrt[3]{3\cdot a}$ $= 2a\sqrt[3]{3\cdot a}$	$\sqrt[3]{24a^4}$ $= \left(24a^4\right)^{\frac{1}{3}}$ $= (24)^{\frac{1}{3}}\cdot(a^4)^{\frac{1}{3}}$ $= (2^3\cdot 3)^{\frac{1}{3}}\cdot\left(a^3\cdot a\right)^{\frac{1}{3}}$ $= 2^{3\cdot\frac{1}{3}}\cdot 3^{\frac{1}{3}}\cdot a^{3\cdot\frac{1}{3}}\cdot a^{\frac{1}{3}}$ $= 2\cdot a\cdot 3^{\frac{1}{3}}\cdot a^{\frac{1}{3}}$ $= 2a\left(3a\right)^{\frac{1}{3}}$ $= 2a\sqrt[3]{3a}$

You Try:

$$\sqrt[4]{96x^5y^4}$$

Definition	Inverse Operation	Fractional Exponent
$\sqrt[4]{96x^5y^4}$ $= \sqrt[4]{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 3\cdot x\cdot x\cdot x\cdot x\cdot x\cdot y\cdot y\cdot y\cdot y}$ $= \sqrt[4]{\underline{2\cdot 2\cdot 2\cdot 2}\cdot 2\cdot 3\cdot \underline{x\cdot x\cdot x\cdot x}\cdot x\cdot \underline{y\cdot y\cdot y\cdot y}}$ $= 2xy\sqrt[4]{6x}$	$\sqrt[4]{96x^5y^4}$ $= \sqrt[4]{2^4\cdot 2\cdot 3\cdot x^4\cdot x\cdot y^4}$ $= \sqrt[4]{2^4\cdot x^4\cdot y^4\cdot 2\cdot 3\cdot x}$ $= \sqrt[4]{2^4\cdot x^4\cdot y^4}\cdot\sqrt[4]{2\cdot 3\cdot x}$ $= 2xy\sqrt[4]{6x}$	$\sqrt[4]{96x^5y^4}$ $= \left(96x^5y^4\right)^{\frac{1}{4}}$ $= (2^4\cdot 2\cdot 3\cdot x^4\cdot x\cdot y^4)^{\frac{1}{4}}$ $= 2^{4\cdot\frac{1}{4}}\cdot 2^{\frac{1}{4}}\cdot 3^{\frac{1}{4}}\cdot x^{4\cdot\frac{1}{4}}\cdot x^{\frac{1}{4}}\cdot y^{4\cdot\frac{1}{4}}$ $= 2\cdot x\cdot y\cdot 2^{\frac{1}{4}}\cdot 3^{\frac{1}{4}}\cdot x^{\frac{1}{4}}$ $= 2xy\left(6x\right)^{\frac{1}{4}}$ $= 2xy\sqrt[4]{6x}$

Students in algebra 2 need to be proficient in fractional exponents for their work in logarithms.

Example 3 Perform the operations and simplify

$$\frac{\left(2x^2\right)^{\frac{3}{2}}}{2^{\frac{1}{2}}x^4}$$

Change to radical form and simplify

$$\frac{\left(2x^2\right)^{\frac{3}{2}}}{2^{\frac{1}{2}}x^4}$$
$$=\frac{\sqrt[2]{\left(2x^2\right)^3}}{\sqrt[2]{2}\cdot x^4}$$
$$=\frac{\sqrt[2]{2^2\cdot 2\cdot x^2\cdot x^2\cdot x^2}}{\sqrt[2]{2}\cdot x^4}$$
$$=\frac{2\cdot x\cdot x\cdot x\sqrt[2]{2}}{\sqrt[2]{2}\cdot x\cdot x\cdot x\cdot x}$$
$$=\frac{2\cdot x\cdot x\cdot x\cdot\sqrt[2]{2}}{\sqrt[2]{2}\cdot x\cdot x\cdot x\cdot x}$$
$$=\frac{2}{x}$$

Keep in exponential form and simplify

$$\frac{\left(2x^2\right)^{\frac{3}{2}}}{2^{\frac{1}{2}}x^4}$$
$$=\frac{2^{\frac{3}{2}}\cdot x^{2\cdot\frac{3}{2}}}{2^{\frac{1}{2}}x^4}$$
$$=\frac{2^{\frac{2}{2}+\frac{1}{2}}\cdot x^3}{2^{\frac{1}{2}}x^4}$$
$$=\frac{2\cdot 2^{\frac{1}{2}}\cdot x^3}{2^{\frac{1}{2}}x^3\cdot x}$$
$$=\frac{2\cdot 2^{\frac{1}{2}}\cdot x^3}{2^{\frac{1}{2}}x^3\cdot x}$$
$$=\frac{2}{x}$$

You Try:

$$\frac{x^{\frac{4}{3}}y^{\frac{2}{3}}}{(xy)^{\frac{1}{3}}}$$

Change to radical form and simplify

$$\frac{x^{\frac{4}{3}}y^{\frac{2}{3}}}{(xy)^{\frac{1}{3}}}$$
$$=\frac{\sqrt[3]{x^4}\cdot\sqrt[3]{y^2}}{\sqrt[3]{xy}}$$
$$=\frac{\sqrt[3]{x^3\cdot x}\cdot\sqrt[3]{y}\cdot\sqrt[3]{y}}{\sqrt[3]{x}\cdot\sqrt[3]{y}}$$
$$=\frac{x\sqrt[3]{x}\cdot\sqrt[3]{y}\cdot\sqrt[3]{y}}{\sqrt[3]{x}\cdot\sqrt[3]{y}}$$
$$=\frac{x\sqrt[3]{x}\cdot\sqrt[3]{y}\cdot\sqrt[3]{y}}{\sqrt[3]{x}\cdot\sqrt[3]{y}}$$
$$=x\sqrt[3]{y}$$

Keep in exponential form and simplify

$$\frac{x^{\frac{4}{3}}y^{\frac{2}{3}}}{(xy)^{\frac{1}{3}}}$$
$$=\frac{x^{\frac{3}{3}+\frac{1}{3}}y^{\frac{1}{3}+\frac{1}{3}}}{x^{\frac{1}{3}}y^{\frac{1}{3}}}$$
$$=\frac{x\cdot x^{\frac{1}{3}}\cdot y^{\frac{1}{3}}\cdot y^{\frac{1}{3}}}{x^{\frac{1}{3}}y^{\frac{1}{3}}}$$
$$=xy^{\frac{1}{3}}$$
$$=x\sqrt[3]{y}$$

*Have students come up and show the different ways that they worked out the simplification. Many ways to do this problem

With rational exponents you can change the index of the root

<table>
<tr><td colspan="2">Example 4 Reduce the index of the radical
$$\sqrt[6]{x^3}$$</td></tr>
<tr><td>$\sqrt[6]{x^3}$
$= x^{\frac{3}{6}}$
$= x^{\frac{1}{2}}$
$= \sqrt[2]{x}$</td><td>You Try: $\sqrt[6]{(x+1)^4}$
$\sqrt[6]{(x+1)^4}$
$= (x+1)^{\frac{4}{6}}$
$= (x+1)^{\frac{2}{3}}$
$= \sqrt[3]{(x+1)^2}$</td></tr>
</table>

<table>
<tr><td colspan="3">Example 5 Use rational exponents to simplify the following powers.</td></tr>
<tr><td>a.
$\sqrt[3]{\sqrt[2]{125}}$
$= \left((125)^{\frac{1}{2}}\right)^{\frac{1}{3}}$
$= \left(5^3\right)^{\frac{1}{6}}$
$= (5)^{3\cdot\frac{1}{6}}$
$= (5)^{\frac{3}{6}}$
$= (5)^{\frac{1}{2}}$
$= \sqrt[2]{5}$</td><td>b.
$\sqrt[4]{\frac{\sqrt[2]{x}}{\sqrt[3]{x}}}$
$= \left(\frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}}\right)^{\frac{1}{4}}$
$= \frac{x^{\frac{1}{8}}}{x^{\frac{1}{12}}}$
$= x^{\frac{1}{8}-\frac{1}{12}}$
$= x^{\frac{1}{24}}$
$= \sqrt[24]{x}$</td><td>c. You Try
$\sqrt[2]{\sqrt[3]{10a^7b}}$
$= \left(\left(10a^7b\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}$
$= \left(10a^7b\right)^{\frac{1}{6}}$
$= \left(10a^6ab\right)^{\frac{1}{6}}$
$= (10ab)^{\frac{1}{6}}\left(a^6\right)^{\frac{1}{6}}$
$= a\sqrt[6]{10ab}$</td></tr>
</table>

Earlier it was stated that algebra 2 students need rational exponents for their work with logarithmic function. The following example is a preview of one type of problem involving logs. In algebra 2, students will learn the definition of logarithmic functions with base *a*. It will be defined as follows:

For $x > 0$, $a > 0$ and $a \neq 1$. $y = \log_a x$ if and only if $x = a^y$

Example 6 Find the exact value of the logarithmic expression without a calculator		
a.	**b.**	**c. You Try**
$y=\log_2\sqrt[4]{8}$ $y=\log_2 8^{\frac{1}{4}}$ $2^y=8^{\frac{1}{4}}$ Apply definition of logarithmic function $2^y=2^{2\cdot\frac{1}{4}}$ $2^y=2^{\frac{1}{2}}$ $y=\frac{1}{2}$ The one to one property allows us to find the value of *y*	$y=\log_5\sqrt[3]{5}$ $y=\log_5 5^{\frac{1}{3}}$ $5^y=5^{\frac{1}{3}}$ $y=\frac{1}{3}$	$y=\log_4\sqrt[3]{16}$ $y=\log_4 16^{\frac{1}{3}}$ $4^y=16^{\frac{1}{3}}$ $4^y=4^{2\cdot\frac{1}{3}}$ $4^y=4^{\frac{2}{3}}$ $y=\frac{2}{3}$

Example 7 Solve for *x*	
a. (one way)	**b. You Try**
$\log_{x+2}4=\frac{2}{3}$ $(x+2)^{\frac{2}{3}}=4$ Apply definition of logarithmic function $(x+2)^{\frac{2}{3}\cdot\frac{3}{2}}=4^{\frac{3}{2}}$ $x+2=\left(4^3\right)^{\frac{1}{2}}$ $x+2=64^{\frac{1}{2}}$ $x+2=8$ $x=6$	$\log_4(x+3)=\frac{3}{2}$ $4^{\frac{3}{2}}=(x+3)$ Apply definition of logarithmic function $\left(4^3\right)^{\frac{1}{2}}=x+3$ $64^{\frac{1}{2}}=x+3$ $8=x+3$ $x+3=8$ $x=5$